Chapter - 5

Continuity and Differentiability

Sol 1:- (i) $f(x) = \begin{cases} 2x+3, & \text{if } x \le 2 \\ 2x-3, & \text{if } x > 2 \end{cases}$

For $x<2$ and $x>2$, $f(x)$ is a polynomial function

$\therefore$ $f(x)$ is continuous except possible at $x=2$

At $x=2$

$$L.H.L = \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 2x+3$$

$$= \lim_{h\to 0} 2(2-h)+3 = 7$$

$$RHL = \lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 2x-3$$

$$= \lim_{h\to 0} 2(2+h)-3 = 1$$

$LHL \neq RHL$

$\therefore$ f is discontinuous at $x=2$

(ii) $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

for $x<0$ and $x>0$ $f(x) = \frac{|x|}{x}$ is continuous

At $x=0$ $LHL = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{-x}{x} = -1$

$$RHL = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{x} = 1$$

$LHL \neq RHL$

$\therefore$ f is discontinuous at $x=0$

(iii) $f(x) = \begin{cases} x^3 - 3 & , \text{ if } x \le 2 \\ 2x - 3 & , \text{ if } x > 2 \end{cases}$

Being a polynomial function for $x < 2$ and $x > 2$, f is continuous.

At $x = 2$ $\quad$ $LHL = \lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} x^3 - 3$

$= \lim\limits_{h \to 0} (2-h)^3 - 3 = 8 - 3 = 5$

$RHL = \lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} x^2 + 1$

$= \lim\limits_{h \to 0} (2+h)^2 + 1 = 4 + 1 = 5$

Also $f(2) = 2^3 - 3 = 8 - 3 = 5$

$LHL = RHL = f(2)$

Hence f is continuous.

(iv) $f(x) = \begin{cases} x^{10} - 1 & , \text{ if } x \le 1 \\ -x^2 & , \text{ if } x > 1 \end{cases}$

For $x < 1$ and $x > 1$, $f(x)$ is a polynomial function

$\therefore$ f is continuous.

At $x = 1$ $\quad$ $LHL = \lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^{10} - 1)$

$= \lim\limits_{h \to 0} (1-h)^{10} - 1 = 1 - 1 = 0$

$RHL = \lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} x^2$

$= \lim\limits_{h \to 0} (1+h)^2 = 1$

$LHL \ne RHL$

$\therefore$ f is discontinuous at $x = 1$

(v) $f(x) = \begin{cases} 3 & \text{if } 0 \le x \le 1 \\ 4 & \text{if } 1 < x < 3 \\ 5 & \text{if } 3 \le x \le 10 \end{cases}$

$f(x)$ is defined in the interval $[0, 10]$

At $x=0$ $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3 = 3 = f(0)$

$\therefore$ f is continuous at $x=0$

Now for $x<1$ and $x>1$

$f(x)$ is a continuous function

$\therefore$ f is continuous

At $x=1$:

$LHL = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3 = 3$

$RHL = \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 4 = 4$

$LHL \ne RHL$

$\therefore$ f is discontinuous at $x=1$

Now for $x<3$ and $x>3$, it is again a constant function and is continuous.

At $x=3$ $LHL \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} 4 = 4$

$RHL = \lim_{x\to 3^+} 5 = 5$

$LHL \ne RHL$

$\therefore$ f is discontinuous at $x=3$

At $x=10$ $\lim_{x\to 10^-} f(x) = \lim_{x\to 10^-} 5 = 5 = f(10)$

$\therefore$ f is continuous at $x=10$

$$f(x) = \begin{cases} 2x & , \text{ if } x < 0 \\ 0 & \text{ if } 0 \le x \le 1 \\ 4x & , x > 1 \end{cases}$$

For $x < 0$, $f(x)$ is a polynomial function and is continuous, for $x > 0$, $f(x)$ is a constant function and is continuous.

At $x = 0$ $\quad LHL = \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2x$

$= \lim_{h \to 0} 2(0 - h) = 0$

$RHL = \lim_{x \to 0^+} (0) = 0$

Also $f(0) = 0$. Since $LHL = RHL = f(0)$

$\therefore$ f is continuous at $x = 0$

Now for $x < 1$, $f(x)$ is a constant function and is continuous, for $x > 1$, $f(x)$ is polynomial function and is continuous

At $x = 1$ $\quad LHL = \underset{x \to 1^-}{Lt} f(x) = \underset{x \to 1^-}{Lt} (0) = 0$

$RHL = \underset{x \to 1^+}{Lt} f(x) = \underset{x \to 1^+}{Lt} (4x)$

$= \underset{h \to 0}{Lt} \; 4(1 + h) = 4$

$LHL \neq RHL$

$\therefore$ f is discontinuous at $x = 1$

Sol. 2:- f will be continuous at $x = 3$.

if $\underset{x \to 3^-}{Lt} f(x) = \underset{x \to 3^+}{Lt} f(x) = f(3)$

$\Rightarrow \underset{x \to 3^-}{Lt} (ax + 1) = \underset{x \to 3^+}{Lt} (bx + 3) = 3a + 1$

Taking $\underset{x \to 3^+}{Lt} (bx + 3) = 3a + 1$

We have $\underset{h \to 0}{Lt} [b(3 + h) + 3] = 3a + 1$

$\Rightarrow \quad 3b + 3 = 3a + 1$

$3b + 2 = 3a$

$\therefore \quad a = b + \frac{2}{3}$

is required relation

Sol. 3:- (i) Given that f is continuous at $x = \frac{\pi}{2}$

$\therefore \lim_{x \to \frac{\pi}{2}} f(x) = f(\frac{\pi}{2})$

$\Rightarrow \lim_{x \to \frac{\pi}{2}} \frac{k \cos x}{\pi - 2x} = 3$

$\Rightarrow \lim_{h \to 0} \frac{k \cos(\frac{\pi}{2} - h)}{1 - 2(\frac{\pi}{2} - h)} = 3$

$\Rightarrow \lim_{h \to 0} \frac{k \sin h}{\pi - \pi + 2h} = \lim_{h \to 0} \frac{k \sin h}{2h} = 3$

$\Rightarrow \frac{k}{2} \times 1 = 3 \quad \therefore k = 6$

(ii) f is continuous at $x = 2$ (given)

$\Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = f(2)$

Taking $\lim_{x \to 2^+} f(x) = f(2)$

we have $\lim_{x \to 2^+} (3) = k(2)^2$

$\Rightarrow 3 = 4k$

$\therefore k = \frac{3}{4}$

(iii) f is continuous at $x = \pi$ (Given)

$\therefore \lim_{x \to \pi^-} f(x) = \lim_{x \to \pi^+} f(x) = f(\pi)$

Taking $\lim_{x \to \pi^+} f(x) = f(\pi)$

we have $\lim_{x \to \pi^+} \cos x = k\pi + 1$

$\Rightarrow \lim_{h \to 0} \cos(\pi + h) = k\pi + 1$

$\Rightarrow \lim_{h \to 0} (-\cos h) = \pi \cdot k + 1$

$\Rightarrow -1 = \pi k + 1$

$\Rightarrow k = -2/\pi$.

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(iv) f is given continuous at $x=5$

$\therefore \lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = f(5)$

Taking $\lim_{x\to 5^+} f(x) = f(5)$

we have $\lim_{x\to 5^+} (3x-5) = 5k+1$

$\Rightarrow \lim_{h\to 0^+} [3(5+h)-5] = 5k+1$

$\Rightarrow 15+3h-5 = 5k+1$

$15-5 = 5k+1$

$9 = 5k$

$k = \frac{9}{5}$

(v): f is continuous at $x=2$

$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2)$

$\Rightarrow \lim_{x\to 2^-} (5) = \lim_{x\to 2^+} [ax+b] = 5$

$\Rightarrow 5 = \lim_{h\to 0} [a(2+h)+b] = 5$

$\Rightarrow 2a+b=5$ — (1)

Also f is continuous at $x=10$

$\therefore \lim_{x\to 10^-} f(x) = \lim_{x\to 10^+} f(x) = f(10)$

$\Rightarrow \lim_{x\to 10^-} (ax+b) = 21$

$\Rightarrow \lim_{h\to 0} [a(10-h)+b] = 21$

$\Rightarrow 10a+b = 21$ — (2)

solving (1) and (2) we get

$a=2$

$b=1$

Sol. 4 :- (i) $x^2 + xy + y^2 = 100$

Diff w.r.t x

$$2x + x\frac{dy}{dx} + y \times 1 = 2y\frac{dy}{dx} = 0$$

$$\Rightarrow (x+2y)\frac{dy}{dx} = -2x - y$$

$$\therefore \frac{dy}{dx} = -\left[\frac{2x+y}{x+2y}\right]$$

(ii) $y = \sin^{-1}\left[\frac{2x}{1+x^2}\right]$, let $x = \tan a$

$$\therefore y = \sin^{-1}\left[\frac{2\tan a}{1+\tan^2 a}\right]$$

$$= \sin^{-1}(\sin 2a) = 2a$$

$$\Rightarrow y = 2\tan^{-1}x$$

$$\therefore \frac{dy}{dx} = \frac{2}{1+x^2}$$

(iii) $y = \cos^{-1}\left[\frac{2x}{1+x^2}\right]$ let $x = \tan a$

$$= \cos^{-1}\left(\frac{2\tan a}{1+\tan^2 a}\right)$$

$$= \cos^{-1}(\sin 2a) = \cos^{-1}\left[\cos\left(\frac{\pi}{2} - 2a\right)\right]$$

$$\Rightarrow y = \frac{\pi}{2} - 2a = \frac{\pi}{2} - 2\tan^{-1}x$$

$$\therefore \frac{dy}{dx} = 0 - \left(\frac{2}{1+x^2}\right) = \frac{-2}{1+x^2}$$

(iv) $y = \sin^{-1}\left[\frac{1-x^2}{1+x^2}\right]$ let $x = \tan a$

$$y = \sin^{-1}\left[\frac{1-\tan^2 a}{1+\tan^2 a}\right]$$

$$= \sin^{-1}(\cos 2a) = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - 2a\right)\right]$$

$$\Rightarrow y = \frac{\pi}{2} - 2a = \frac{\pi}{2} - 2\tan^{-1}x$$

$$\therefore \frac{dy}{dx} = 0 - \left(\frac{2}{1+x^2}\right) = \frac{-2}{1+x^2}$$

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(v) $y = \sin^{-1}(2x\sqrt{1-x^2})$ put $x = \sin\theta$

$= \sin^{-1}(2\sin\theta\sqrt{1-\sin^2\theta})$

$\Rightarrow y = 2\theta = 2\sin^{-1}x$

$\therefore \dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-x^2}}$

Sol: 5 : (i) Differentiate $\log(\cos e^x)$

$y = \log[\cos(e^x)]$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos(e^x)} \times [-\sin(e^x)] \times e^x$

$= -e^x \tan(e^x)$

(ii) $y = e^x + e^{x^2} + e^{x^3} + e^{x^4} + e^{x^5}$

$\Rightarrow \dfrac{dy}{dx} = e^x + 2x \cdot e^{x^2} + 3x^2 e^{x^3} + 4x e^{x^4} + 5x^4 e^{x^5}$

(iii) $y = \dfrac{\cos x}{\log x}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{\log x(-\sin x) - \cos x \times \frac{1}{x}}{(\log x)^2}$

$= \dfrac{-[x\sin x \log x + \cos x]}{x(\log x)^2}$

Sol: 6 (i) Let $y = \cos x \cos 2x \cos 3x$

$\Rightarrow \log y = \log\cos x + \log\cos 2x + \log\cos 3x$

Diff. w.r.t. x

$\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{\cos x}(-\sin x) + \dfrac{1}{\cos 2x} \times (-\sin 2x) \times 2$

$+ \dfrac{1}{\cos 3x} \times \sin 3x \times 3$

$\Rightarrow \dfrac{dy}{dx} = -y[\tan x + 2\tan 2x + 3\tan 3x]$

$\therefore \dfrac{dy}{dx} = -\cos x \cos 2x \cos 3x(\tan x + 2\tan 2x$

$+ 3\tan 3x)$

(ii) Diff. $\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$ w.r.t x

$\Rightarrow$ ~~log~~ let $y = \sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

$\Rightarrow \log y = \frac{1}{2}[\log(x-1) + \log(x-2) - \log(x-3) - \log(x-4) - \log(x-5)]$

Diff. w.r.t x.

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$

$\therefore \frac{dy}{dx} = \frac{1}{2}\sqrt{\dfrac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1} + \frac{1}{x-2} - \frac{1}{x-3} - \frac{1}{x-4} - \frac{1}{x-5}\right]$

(iii) Let $y = (\log x)^{\cos x}$

$\Rightarrow \log y = \cos x \log x$

Diff w.r.t x

$\frac{1}{y}\frac{dy}{dx} = \cos x \times \frac{1}{\log x} \times \frac{1}{x} + [\log(\log x)][-\sin x]$

$\Rightarrow \frac{dy}{dx} = y\left[\frac{\cos x}{\log x} - \sin x \log(\log x)\right]$

$\therefore \frac{dy}{dx} = (\log x)^{\cos x}\left[\frac{\cos x}{\log x} - \sin x \log(\log x)\right]$

(iv) Let $y = x^x - 2^{\sin x} = A - B$

$\Rightarrow \frac{dy}{dx} = \frac{dA}{dx} - \frac{dB}{dx}$.—(1)

where $A = x^x$

$\Rightarrow \log A = x \log x$

Diff w.r.t x

$\frac{1}{A}\frac{dA}{dx} = x \times \frac{1}{x} + \log x(1)$

$B = 2^{\sin x}$

$\Rightarrow \frac{dB}{dx} = 2^{\sin x} \cos x \times \log 2$

$= \cos x (2^{\sin x}) \log 2$

$\Rightarrow \log A = x\log x$

$\Rightarrow \dfrac{dA}{dx} = A(1+\log x) = x^x(1+\log x)$

$\dfrac{dy}{dx} = \dfrac{dA}{dx} - \dfrac{dB}{dx}$

$= x^x(1+\log x) - \cos x(2^{\sin x})\log 2$

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(v) Let $y = (\log x)^x + x^{\log x}$

$\Rightarrow y = A + B \Rightarrow \dfrac{dy}{dx} = \dfrac{dA}{dx} + \dfrac{dB}{dx}$ —(1)

where $A = (\log x)^x \Rightarrow \log A = x\log(\log x)$

Diff w.r.t. x,

$\dfrac{1}{A}\dfrac{dA}{dx} = x \times \dfrac{1}{\log x} \times \dfrac{1}{x} + \log(\log x) \times 1$

$\Rightarrow \dfrac{dA}{dx} = A\left[\dfrac{1}{\log x} + \log(\log(x))\right]$

$= (\log x)^x\left[\dfrac{1}{\log x} + \log(\log x)\right]$

and $B = x^{\log x}$

$\Rightarrow \log B = \log x . \log x = (\log x)^2$

Diff. w.r.t. x

$\dfrac{1}{B}\dfrac{dB}{dx} = 2(\log x) \times \dfrac{1}{x}$

$\dfrac{dB}{dx} = B\left(\dfrac{2\log x}{x}\right) = x^{\log x}\left(\dfrac{2\log x}{x}\right)$

$= 2x^{\log x - 1} . \log x$

$= \dfrac{dy}{dx} = (\log x)^x\left[\dfrac{1}{\log x} + \log(\log x)\right] + 2x^{\log x - 1}\log x$

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(vi) Let $y = \left(x+\frac{1}{x}\right)^x + x^{\left(1+\frac{1}{x}\right)}$

$\Rightarrow \quad y = A + B \Rightarrow \frac{dy}{dx} = \frac{dA}{dx} + \frac{dB}{dx} \quad -(1)$

where $A = \left(x+\frac{1}{x}\right)^x \Rightarrow \log A = x \log\left(x+\frac{1}{x}\right)$

Diff. w.r.t x

$$\frac{1}{A}\frac{dA}{dx} = x \times \frac{1}{x+\frac{1}{x}}\left[1-\frac{1}{x^2}\right] + \log\left(x+\frac{1}{x}\right] \times 1$$

$$\Rightarrow \frac{dA}{dx} = A\left[\left(\frac{x^2}{x^2+1}\right)\left(\frac{x^2-1}{x^2}\right) + \log\left(x+\frac{1}{x}\right)\right]$$

$$= \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right]$$

and $B = x^{\left(1+\frac{1}{x}\right)} \Rightarrow \log B = \left(1+\frac{1}{x}\right)\log x$

Diff. w.r.t. x

$$\frac{1}{B}\frac{dB}{dx} = \left(1+\frac{1}{x}\right)\times\frac{1}{x} + \log\left[0 - \frac{1}{x^2}\right]$$

$$\Rightarrow \frac{dB}{dx} = x^{\left(1+\frac{1}{x}\right)}\left(\frac{1+x-\log x}{x^2}\right)$$

$$\Rightarrow \frac{dy}{dx} = \left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1} + \log\left(x+\frac{1}{x}\right)\right] + x^{\left(1+\frac{1}{x}\right)}\left(\frac{1+x-\log x}{x^2}\right)$$

Sol 27 (i) Let $x^y + y^x = 1$

$\Rightarrow \quad A + B = 1 \quad \Rightarrow \frac{dA}{dx} + \frac{dB}{dx} = 0 \quad -(1)$

where $A = x^y$ $\qquad B = y^x$

$\Rightarrow \log A = y \log x$ $\qquad \Rightarrow \log B = x \log y$

Diff w.r.t x

$$\frac{1}{A}\frac{dA}{dx} = \frac{y}{x} + \log x \frac{dy}{dx} \qquad \frac{1}{B}\frac{dB}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y \times 1$$

$$\Rightarrow \frac{dA}{dx} = A\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) \qquad \Rightarrow \frac{dB}{dx} = B\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$

$\Rightarrow \frac{dA}{dx} = x^y\left(\frac{y}{x} + \log x \frac{dy}{dx}\right)$ $\Rightarrow \frac{dB}{dx} = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$

(i) $\Rightarrow \frac{dA}{dx} + \frac{dB}{dx} = 0$

$\Rightarrow x^y\left(\frac{y}{x} + \log x \frac{dy}{dx}\right) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0$

$\Rightarrow x^y\left(\frac{y}{x} + \log x \frac{dy}{dx}\right) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0$

$\Rightarrow \left[x^y \log x + y^{x-1} \cdot x\right]\frac{dy}{dx} = -\left[x^{y-1} \cdot y + y^x \log y\right]$

$\therefore \frac{dy}{dx} = -\left[\frac{x^{y-1} \cdot y + y^x \log y}{x^y \log x + y^{x-1} x}\right]$

(ii) $y^x = x^y \Rightarrow x \log y = y \log x$

Diff. w.r.t x

$x \times \frac{1}{y}\frac{dy}{dx} + \log y \times 1$

$= y \times \frac{1}{x} + \log x \frac{dy}{dx}$

$\Rightarrow \left(\frac{x}{y} - \log x\right)\frac{dy}{dx} = \left(\frac{y}{x} - \log y\right)$

$\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$

(iii) $(\cos x)^y = (\cos y)^x \Rightarrow y \log \cos x = x \log \cos y$

$= \cancel{x \times \frac{1}{\cos y}}$

Diff. w.r.t. x $y \times \frac{1}{\cos x}(-\sin x) + (\log \cos x)\frac{dy}{dx}$

$= x \times \frac{1}{\cos y}(\sin y)\frac{dy}{dx} + \log \cos y$

$$\Rightarrow \left(\log\cos x + x\tan y\right)\frac{dy}{dx} = \log\cos y + y\tan x$$

$$\therefore \frac{dy}{dx} = \frac{\log\cos y + y\tan x}{\log\cos x + x\tan y}$$

(iv) $$xy = e^{x-y}$$

$$\Rightarrow \log x + \log y = x - y$$

Diff w.r.t. x

$$\frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 1 - \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y}+1\right)\frac{dy}{dx} = 1 - \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$$

Sol: 8 :- (i) $x = \sin t$

$$\Rightarrow \frac{dx}{dt} = \cos t$$

$y = \cos 2t$

$$\frac{dy}{dt} = -\sin 2t \times 2$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{-2\sin 2t}{\cos t}$$

$$= \frac{-2\times 2\sin t\cos t}{\cos t} = -4\sin t$$

(ii) $$x = \frac{\sin^3 t}{\sqrt{\cos 2t}}$$

$$\Rightarrow \frac{dx}{dt} = \frac{\sqrt{\cos 2t}\times 3\sin^2 t\cos t - \sin^3 t\times\frac{-\sin 2t\times 2}{2\sqrt{\cos 2t}}}{\cos 2t}$$

$$= \frac{3\cos t\cos 2t\sin^2 t + \sin 2t\sin^3 t}{(\cos 2t)^{3/2}}$$

and $$y = \frac{\cos^3 t}{\sqrt{\cos 2t}}$$

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$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{\cos 2t}\; 3\cos^2 t(-\sin t) - \cos^3 t \dfrac{(-\sin 2t \times 2)}{2\sqrt{\cos 2t}}}{\cos 2t}$$

$$= \frac{-3\sin t\cos^2 t(\cos 2t) + \sin 2t\cos^3 t}{(\cos 2t)^{3/2}}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx}$$

$$= \frac{-3\sin t\cos^2 t\cos 2t + \sin 2t\cos^3 t}{3\cos t\cos 2t\sin^2 t + \sin 2t\sin^3 t}$$

$$= \frac{\sin 2t\left(-\frac{3}{2}\cos t\cos 2t + \cos^3 t\right)}{\sin 2t\left(\frac{3}{2}\sin t\cos 2t + \sin^3 t\right)}$$

$$= \frac{-\frac{3}{2}\cos t\cos 2t + \cos^3 t}{\frac{3}{2}\sin t\cos 2t + \sin^3 t}$$

$$= \frac{-\frac{3}{2}\cos t(2\cos^2 t - 1) + \cos^3 t}{\frac{3}{2}\sin t(1 - 2\sin^2 t) + \sin^3 t}$$

$$= \frac{-2\cos^3 t + \frac{3}{2}\cos t}{\frac{3}{2}\sin t - 2\sin^3 t}$$

$$= \frac{-(4\cos^3 t - 3\cos t)}{3\sin t - 4\sin^3 t}$$

$$= -\frac{\cos 3t}{\sin 3t} = -\cot 3t$$

Sol: 9:- (i) 2^{nd} order derivatives

$y = e^x \sin 5x$

$y_1 = 5 \times e^x \cos 5x + \sin 5x \cdot e^x$

$= 5e^x \cos 5x + e^x \sin 5x$

$\Rightarrow y_2 = 5[e^x(-\sin 5x) \times 5 + \cos 5x \cdot e^x]$

$\cancel{+[5e^x \cos 5x + \sin 5x\, e^x]}$

$+ [e^x(\cos 5x) \cdot 5 + \sin 5x\, e^x]$

$= -25e^x \sin 5x + 5e^x \cos 5x + 5e^x \cos 5x + e^x \sin 5x$

$\therefore y_2 = 10e^x \cos 5x - 24e^x \sin 5x$

$= 2e^x(5\cos 5x - 12\sin 5x)$

(ii) $y = x^3 \log x$

$\Rightarrow y_1 = x^3 \frac{1}{x} + (\log x) \cdot 3x^2$

$\Rightarrow y_1 = x^2 + 3x^2 \log x$

$\Rightarrow y_2 = 2x + 3[x^2 \times \frac{1}{x} + (\log x) 2x]$

$\Rightarrow y_2 = 2x + 3(x + 2x \log x)$

$= x[5 + 6\log x]$

Sol: 10:- $y = 5\cos x - 3\sin x$

$\Rightarrow y_1 = -5\sin x - 3\cos x$

$\Rightarrow y_2 = -5\cos x + 3\sin x$

$= -[5\cos x - 3\sin x]$

$\Rightarrow y_2 = -y$

$\therefore y_2 + y = 0$

Sol. 11:- $y = \cos^{-1}x \Rightarrow x = \cos y$

Diff w.r.t y

$$\frac{dx}{dy} = -\sin y$$

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec} y$$

Diff w.r.t x

$$\frac{d^2y}{dx^2} = \operatorname{cosec} y \cot y \frac{dy}{dx}$$

$$= \operatorname{cosec} y \cot y(-\operatorname{cosec} y)$$

$$= -\cot y \operatorname{cosec}^2 y$$

Sol 12:- $y = 3\cos(\log x) + 4\sin(\log x)$

$$\Rightarrow y_1 = \frac{-3\sin(\log x)}{x} + \frac{4\cos(\log x)}{x}$$

$\Rightarrow xy_1 = -3\sin(\log x) + 4\cos(\log x)$

Diff again w.r.t x

$$xy_2 + y_1 = -\frac{3\cos(\log x)}{x} - 4\sin(\log x) \times \frac{1}{x}$$

$\Rightarrow x^2y_2 + xy_1 = -[3\cos(\log x) + 4\sin(\log x)] = -y$

$\therefore x^2y_2 + xy_1 + y = 0$

Sol 13:-

$$y = Ae^{mx} + Be^{nx}$$

$$\Rightarrow y_1 = mAe^{mx} + nBe^{nx}$$

$$\Rightarrow y_2 = m^2Ae^{mx} + n^2Be^{nx}$$

$$L.H.S = y_2 - (m+n)y_1 + mny$$

$$= m^2Ae^{mx} + n^2Be^{nx} - (m+n)(mAe^{mx} + Be^{nx})$$
$$mn(Ae^{mx} + Be^{nx})$$

$$= m^2Ae^{mx} + n^2Be^{nx} - m^2Ae^{mx} - mnBe^{nx}$$
$$- mne^{mx} - n^2Be^{nx} + mnAe^{mx}$$
$$+ mnBe^{nx} = 0 = R.H.S$$

Sol.14:-

$$y = 500e^{7x} + 600e^{-7x}$$

$$\Rightarrow y_1 = 3500e^{7x} - 4200e^{-7x}$$

$$\Rightarrow y_2 = (7\times3500)e^{7x} + (7\times4200)e^{-7x}$$

$$= 49(500e^{7x} + 600)e^{-7x}] = 49y$$

$\therefore$ $y_2 = 49y$, hence the result.

Sol.15:- $y = (\tan^{-1}x)^2$ — (1)

$$\Rightarrow y_1 = 2(\tan^{-1}x)\times\frac{1}{1+x^2}$$

$$\Rightarrow (1+x^2)y_1 = 2(\tan^{-1}x)$$

squaring both sides

$$(1+x^2)^2y_1^2 = 4y \quad \text{-- using (1)}$$

Diff w.r.t x

$$(1+x^2)^2 \cdot 2y_1y_2 + y_1^2\times2(1+x^2)\cdot2x = 4y_1$$

Dividing both side by $2y_1$, we get

$$(1+x^2)^2y_2 + 2x(x^2+1)y_1 = 2$$

Hence proved.

Sol. 16 :- $f(x) = x^2+2x-8 \Rightarrow f'(x) = 2x+2$ [Rolles Theorem]

1. Being a polynomial function, $f(x)$ is Continuous on $[-4,2]$
2. $f'(x)$ exist uniquely on $(-4,2)$
   $\therefore f(x)$ is derivable on $(-4,2)$
3. $f(-4) = 16-8-8=0$
   $f(2) = 4+4-8=0$
   $\therefore f(-4)=f(2)$

Since $f(x)$ satisfies all the condition of Rolles Theorem there exist at least one $c \in (-4,2)$ s.t

$$f'(c) = 0$$

$$\Rightarrow 2c+2=0 \quad \Rightarrow c=-1 \in (-4,2)$$

Hence the Theorem varified

Sol : 17 :- Mean value Theorem

Given $f(x) = x^2-4x-3$

$\Rightarrow f'(x) = 2x-4$

(i) $f(x)$ is a polynomial function
   $\therefore f(x)$ is continuous on $[1,4]$

(ii) $f'(x)$ exists uniquely in $(1,4)$
   $\therefore f(x)$ is derivable in $(1,4)$

Since $f(x)$ satisfies both condition of L.M.V Theorem $\therefore$ There exist at least one $c \in (1,4)$ s.t

$$f'(c) = \frac{f(4)-f(1)}{4-1}$$

$$\Rightarrow 2c-4 = \frac{(16-16-3)-(1-4-3)}{3}$$

$$\Rightarrow 2c-4 = 1 \quad \Rightarrow c = \frac{5}{2} \in (1,4)$$

Hence theorem varified

Sol: 18 :- Let $y = (\log x)^{\log x}$ (33)

$\Rightarrow \log y = \log x \cdot \log(\log x)$

Diff. w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = (\log x) \times \frac{1}{\log x} \times \frac{1}{x} + \log(\log x) \times \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\frac{1}{x} + \frac{\log(\log x)}{x}\right]$$

$$\therefore \frac{dy}{dx} = (\log x)^{\log x} \cdot \frac{1}{x}\left[1 + \log(\log x)\right]$$